Quantum Channels (Part 1)

Recap:

Quantum (Information) Theory in a slide....

Given a quantum system S, a quantum state is any Hermitian matrix ρ such that:

1. (Non-negativity) $eigs(\rho) \geq 0$.
2. (Normalization) $\mathrm{tr}(\rho) = 1$.

Quantum states

Quantum evolutions

Given a quantum system S in a state ρ any quantum evolution is described by a linear transformation $\rho \to \mathcal{E}(\rho)$, such that $\sigma = \mathcal{E}(\rho)$ is always another valid quantum state.

last part of the wwe story.

Given a quantum system S of dimension d, any general quantum measurement $\mathcal{M}$ on S with m outcomes is described by a set of matrices $\mathcal{M} = \{M_0, M_1, \ldots, M_{m-1}\}$ such that

1. (Non-negativity) $eigs(M_i) \geq 0$ for all $i = 0, 1, \ldots m-1$.
2. (Normalization) $\sum_i M_i = \mathbb{1}$.

Where $\mathbb{1}$ is the $d \times d$ identity matrix. Moreover, given a quantum state ρ, the probability of the k'th outcome of doing the measurement $\mathcal{M}$ on the state is given by

$$\mathrm{Prob}[k] = \mathrm{tr}[M_k \rho]. \tag{16}$$

Quantum measurements

Basic requirements on $\mathcal{E}$ & the Kraus representation

1) linearity: $\mathcal{E}(p\rho + (1-p)\sigma) = p\mathcal{E}(\rho) + (1-p)\mathcal{E}(\sigma)$

Intuition (A) Toss a coin & prepare ρ or σ and then apply $\mathcal{E}$. Output resulting state.

(B) Apply $\mathcal{E}$ to ρ and σ, toss a coin and then output $\mathcal{E}(\rho)$ or $\mathcal{E}(\sigma)$.

Linearity captures the fact that the state output from procedures (A) & (B) are entirely indistinguishable.

2) Output state needs to be a real state

i.e. i. $\mathrm{Tr}(\mathcal{E}(\rho)) = 1$ (normalized)

ii. $\mathcal{E}(\rho) = \mathcal{E}(\rho)^\dagger$ (Hermitian)

iii. $\mathrm{eigs}(\mathcal{E}(\rho)) \geq 0$ (positivity/non-negativity)

These requirements on $\mathcal{E}$ are satisfied if

$$\boxed{\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger \quad \text{s.t.} \quad \sum_i A_i^\dagger A_i = I}$$

In fact this 'if' can be upgraded to an 'only if' if we require that the output of the channel applied to just a subsystem is a real state. Will come back to this later.

This is called the Kraus Representation

Proof

1) $$\mathcal{E}(p\rho + (1-p)\sigma) = \sum_i A_i (p\rho + (1-p)\sigma) A_i^\dagger$$
$$= p \sum_i A_i \rho A_i^\dagger + (1-p) \sum_i A_i \sigma A_i^\dagger$$
$$= p\,\mathcal{E}(\rho) + (1-p)\,\mathcal{E}(\sigma) \quad \checkmark$$

2) i. $$\mathrm{Tr}(\mathcal{E}(\rho)) = \mathrm{Tr}\left(\sum_i A_i \rho A_i^\dagger\right)$$
$$= \sum_i \mathrm{Tr}(A_i \rho A_i^\dagger)$$
$$= \sum_i \mathrm{Tr}(A_i^\dagger A_i \rho)$$
$$= \mathrm{Tr}\left(\sum_i A_i^\dagger A_i \rho\right)$$
$$= \mathrm{Tr}(\rho)$$

trace preserved so if $\mathrm{Tr}(\rho) = 1$ then $\mathrm{Tr}(\mathcal{E}(\rho)) = 1$ $\checkmark$

2) ii. $\text{eigs}(\sigma) \geq 0 \iff \langle\phi|\sigma|\phi\rangle \geq 0 \ \forall |\phi\rangle$

Any normalized quantum state

(to see this write $\sigma = \sum_\lambda \lambda |\lambda\rangle\langle\lambda|$

$$\underbrace{\langle\phi|\sigma|\phi\rangle}_{\geq 0 \text{ iff } \lambda \geq 0} = \sum_\lambda \lambda \underbrace{|\langle\phi|\lambda\rangle|^2}_{\geq 0}$$
)

Showing that $\langle\phi|\, \mathcal{E}(\rho)\, |\phi\rangle \geq 0 \quad \forall |\phi\rangle$

$\sum_i A_i (\ ) A_i^\dagger$

write as $\sum \lambda |\lambda\rangle\langle\lambda|$

$$\langle\phi| \sum_i A_i \rho A_i^\dagger |\phi\rangle = \sum_i \langle\phi| A_i \rho A_i^\dagger |\phi\rangle$$

$$= \sum_{i\lambda} \lambda \langle\phi| A_i |\lambda\rangle\langle\lambda| A_i^\dagger |\phi\rangle$$

$$= \underbrace{\sum_{i\lambda} \lambda \underbrace{|\langle\phi| A_i |\lambda\rangle|^2}_{+ve}}_{\therefore\ +ve \text{ if } \rho\ +ve}$$

✓

Examples of Common Channels

1) Unitary Dynamics $\quad \mathcal{E}(\ ) = U(\ )U^\dagger$

$UU^\dagger = U^\dagger U = I$

Check: $\sum_i A_i^\dagger A_i = U^\dagger U = I$

Covered previously

It's helpful to know how a unitary on a single qubit effects a state on a bloch sphere.

Let $\rho = \frac{1}{2}(I + \underline{r} \cdot \underline{\sigma})$

$$\mathcal{E}(\rho) = U \rho U^\dagger = \tfrac{1}{2}(I + \underline{s} \cdot \underline{\sigma}) \qquad = U(\underline{r} \cdot \underline{\sigma}) U^\dagger$$

Claim: $\underline{s} = O \, \underline{r}$ where O is an orthogonal matrix

Orthogonal matrices induce (length preserving) rotations on real vectors.

$\left\{ \begin{array}{l} \text{ie. } O O^T = O^T O = I \\ \mathrm{Real}(O) = O \end{array} \right.$

(ie. they are the real analogue of unitaries)

Proof:

To show that O is orthogonal we just need to show that $|\underline{r}| = |\underline{s}|$

To do so, we note that
$$\mathrm{Tr}\left((\underline{s} \cdot \underline{\sigma})^2\right) = \mathrm{Tr}\left(\left(\textstyle\sum_i s_i \sigma_i\right)^2\right)$$
$$= \mathrm{Tr}\left(\textstyle\sum_{ij} s_i s_j \sigma_i \sigma_j\right)$$
$$= \textstyle\sum_{ij} s_i s_j \delta_{ij}$$
$$= \textstyle\sum_i s_i^2 = |s|^2$$

Now we note that
$$|s|^2 = \mathrm{Tr}\left((\underline{s} \cdot \underline{\sigma})^2\right) = \mathrm{Tr}\left((U \, \underline{r} \cdot \underline{\sigma} \, U^\dagger)^2\right) = \mathrm{Tr}\left(U (\underline{r} \cdot \underline{\sigma})^2 U^\dagger\right) = \sum_{ij} \mathrm{Tr}\left(r_i r_j U \sigma_i \sigma_j U^\dagger\right)$$
$$= \sum r_i^2 = |r|^2$$

$$\therefore \; |\underline{s}| = |\underline{r}| \implies O \text{ is orthogonal}$$

To get a better handle on the nature of this rotation, note that:

Any unitary on a qubit can be written as

$$U = e^{i\theta\, \underline{n}\cdot\underline{\sigma}}$$

(if this isn't immediately obvious note that any unitary can be written as $U = e^{-iHt}$ and H can always be expanded in the Pauli basis)

which is equivalent to

$$U = e^{i\theta\, \underline{n}\cdot\underline{\sigma}} = \cos(\theta)\, I + i \sin(\theta)\, \underline{n}\cdot\underline{\sigma}$$

(if you've not shown this before it's worth proving it to yourself)

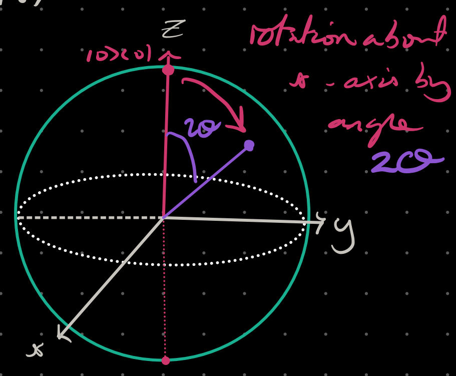
What is the effect of evolving $|0\rangle\langle 0|$ under $e^{-i\theta\sigma_x}$?

$$e^{+i\theta\sigma_x/2}\, |0\rangle = \cos\theta\, |0\rangle + i\sin(\theta)\, \sigma_x\, |0\rangle$$

$$= \cos\theta\, |0\rangle + i\sin\theta\, |1\rangle$$

eg. $$e^{+\frac{\pi}{2}\sigma_x}\, |0\rangle = \cos(\pi/4)\, |0\rangle + i\sin(\pi/4)\, |1\rangle$$

$$= |+_y\rangle$$



Exercise: Show that $e^{-i\theta\, \underline{n}\cdot\underline{\sigma}}$ corresponds to a rotation about $\underline{n}$ vector by an angle 2θ

2) Isometric Dynamics $\mathcal{E}(\ ) = U(\ )U^\dagger$

Check: $\sum_i A_i^\dagger A_i = U^\dagger U = I$

Note we only use $U^\dagger U = I$ & not $UU^\dagger = I$ ∴ U could be an isometry (that isn't also unitary)

eg. consider
$$U|0\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |2\rangle)$$
$$U|1\rangle = \tfrac{1}{\sqrt{2}}(|1\rangle + |3\rangle)$$

This maps the system from a 2d Hilbert space $\{|0\rangle, |1\rangle\}$ to a 4d space $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$

3) Convex combination of Unitaries

$$\mathcal{E}(\ ) = \sum_i p_i U_i(\ )U_i^\dagger$$

$0 \le \ \le 1$ $\quad$ $U_iU_i^\dagger = U_i^\dagger U_i = I$

Read: "apply unitary U_i with probability p_i"

$$A_i = \sqrt{p_i}\, U_i$$

Check: $\sum_i A_i^\dagger A_i = \sum_i p_i U_i^\dagger U_i = \sum_i p_i I = I$ ✓

4) Completely Dephasing: $A_0 = |0\rangle\langle 0|$, $A_1 = |1\rangle\langle 1|$

Check $\sum_i A_i^\dagger A_i = |1\rangle\langle 1| + |0\rangle\langle 0| = I$ ✓

What does this channel do?

$$\rho = \frac{1}{2}(I + \underline{r}\cdot\underline{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

$$\mathcal{E}(\rho) = \langle 0|\rho|0\rangle\, |0\rangle\langle 0| + \langle 1|\rho|1\rangle\, |1\rangle\langle 1|$$

$$= \frac{1}{2}\begin{pmatrix} 1+z & 0 \\ 0 & 1-z \end{pmatrix}$$

$\Rightarrow$ Dephasing map kills the off-diagonal components & returns a classical state.

5) Completely Depolarizing Map

Given some fixed state τ

$$\mathcal{E}(\rho) = \tau \quad \forall \rho$$

This is a valid channel because it's linear & outputs a legitimate state.

So it must be possible to write it in terms of Kraus operators...

To see how first write τ in terms of its eigendecomposition

$$\tau = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

Then we can use: $\quad A_{ij} = \sqrt{\lambda_i}\, |\lambda_i\rangle\langle j|$

To see why this works note that

$$\mathcal{E}(\rho) = \sum_{ij} \lambda_i\, |\lambda_i\rangle\langle j|\rho|j\rangle\langle\lambda_i|$$

$$= \mathrm{Tr}(\rho)\, \tau = \tau \quad ✓$$

Also check $\sum_{ij} A_{ij}^\dagger A_{ij} = \sum_{ij} \lambda_i \, |j\rangle \overset{1}{\cancel{\langle \lambda_i | \lambda_i \rangle}} \langle j|$

$= \sum_i \underset{1}{\cancel{\lambda_i}} \sum_j |j\rangle\langle j| = I$

> **Exercise:** What about a non-completely polarizing map?
>
> eg. one that outputs τ with probability p
>
> & ρ " " $1-p$
>
> What are the Kraus operators now?

6) The "Measure-record-and-update" Channel

$$\mathcal{E}(\rho) = \sum_k \underbrace{|k\rangle\langle k|_X}_{\text{classical register of measurement outcome}} \otimes B_k \, \rho_S \, B_k^\dagger$$

system being measured

$$A_k = |k\rangle \otimes B_k$$

$$\mathcal{E}(\rho) = \sum_k |k\rangle \otimes B_k \,(I \otimes \rho)\, \langle k| \otimes B_k^\dagger$$

$$= \sum_k |k\rangle\langle k| \otimes B_k \, \rho \, B_k^\dagger$$

agree

Check: $\sum_k A_k^\dagger A_k = \sum_k \overset{1}{\cancel{\langle k|k\rangle}} \underbrace{B_k^\dagger B_k = I}_{\text{iff } \sum_k B_k^\dagger B_k}$

i.e. $\{A_k\}$ is a legit set of Kraus operators iff $\{B_k\}$ are a set of Kraus operators.

What has this got to do with measurements?

Let's look at the reduced output state on the classical register

$$\mathrm{Tr}_S(\mathcal{E}(\rho)) = \mathrm{Tr}_S\left(\sum_k |k\rangle\langle k| \otimes B_k \rho B_k^\dagger \right)$$

$$= \sum_k |k\rangle\langle k| \, \mathrm{Tr}_S(B_k \rho B_k^\dagger) \qquad M_k := B_k^\dagger B_k$$

$$= \sum_k \mathrm{Tr}(M_k \rho) \, |k\rangle\langle k| \qquad p_k := \mathrm{Tr}(M_k \rho)$$

$$= \sum_k p_k |k\rangle\langle k|_X$$

Because $M_k^\dagger = (B_k^\dagger B_k)^\dagger = B_k^\dagger B_k = M_k$ is Hermitian it has real eigenvalues. Moreover, these eigenvalues are necessarily non-negative (because $\langle\phi| B_k^\dagger B_k |\phi\rangle = \| B_k|\phi\rangle \|_2^2 \geq 0 \ \forall \phi$)

So M_k is a +ve Hermitian operator

& $\sum_k M_k = \sum_k B_k^\dagger B_k = I$ by assumption

$\therefore \{ M_k \}$ defines a POVM

& p_k $= \mathrm{Tr}(M_k \rho)$ is just the probability of getting the kth outcome

If we measure the classical register, as it's in the state $\sum_k p_k |k\rangle\langle k|$ we get the kth outcome with probability p_k i.e. the measurement outcome has been recorded in the classical register X.

What if we trace out the second register instead?

Then we get

$$\mathrm{Tr}_X\big(\mathcal{E}(\rho)\big) = \mathrm{Tr}_X\Big(\sum_k |k\rangle\langle k|_X \otimes B_k \rho B_k^\dagger\Big)$$

$$= \sum_k B_k \rho B_k^\dagger \quad \Big\} \text{ form of a standard channel}$$

$$\mathcal{E}(\rho) = \sum_k \underbrace{|k\rangle\langle k|_X}_{\text{classical register of measurement outcome}} \otimes B_k \rho_S B_k^\dagger$$

~ measurement update rule (classically correlated with corresponding recorded measurement outcome)

That is, if we look at the state of the system conditioned on obtaining measurement outcome $|x\rangle\langle x|_X$ we have

$$\rho \to \mathcal{E}(\rho) \xrightarrow{\text{"get } x\text{"}} \underbrace{|x\rangle\langle x| \otimes B_x \rho B_x^\dagger}_{\text{i.e. } (|x\rangle\langle x| \otimes I)\,\mathcal{E}(\rho)\,(|x\rangle\langle x| \otimes I)}$$

This is not a normalized state, the normalized output state would be

$$\rho_{out} = |x\rangle\langle x| \otimes \rho_x \quad \text{with} \quad \rho_x = \frac{B_x \rho B_x^\dagger}{\mathrm{Tr}(M_x \rho)}$$

i.e. a POVM $\{M_k\}$ such that $M_k = B_k^\dagger B_k$

tells us two things ① The probability of getting output k

$\mathrm{Tr}(M_x \rho)$

② An update rule

$$\rho \to \rho_x = \frac{B_x \rho B_x^\dagger}{\mathrm{Tr}(M_x \rho)}$$

But note this update rule is not unique.

Say $B_k \to \tilde{B}_k = U B_k$ — some unitary / isometry

$$\tilde{B}_k^\dagger \tilde{B}_k = B_k^\dagger U^\dagger U B_k = B_k^\dagger B_k = M_k$$

Thus it's important to draw a distinction between the POVM, the measurement, which is all about how to extract information from a state & the (optional) state update rule which concerns the evolution of the state.

☆

7) Partial Trace $\mathcal{E}(\rho_{AB}) = \mathrm{Tr}_B(\rho_{AB}) = \rho_B$

$$A_k = I_A \otimes \langle k|_B$$

Check it works $\sum_k A_k \rho A_k^\dagger = \sum_k (I_A \otimes \langle k|)\, \rho\, (I_A \otimes |k\rangle)$

$$:= \mathrm{Tr}_B(\rho) \checkmark$$

$$\sum_k A_k^\dagger A_k = I_A \otimes \sum_k |k\rangle\langle k| = I_{AB} \checkmark$$

8) State Preparation

$$\underset{\text{trivial input}}{\mathbb{1}} \longrightarrow \rho \quad \text{state to be prepare}$$

$$A_k = \sqrt{\lambda_i}\,|\lambda_i\rangle$$

Check it works $\quad \sum_u A_u(\mathbb{1})A_u^\dagger = \sum_u \lambda_u |u\rangle\langle u|$

$$= \rho \quad ✓$$

Check $\quad \sum_u A_u^\dagger A_u = \sum_u \lambda_u \overset{1}{\langle u|u\rangle} = \mathbb{1}$

Note:

Combinations of channels are channels

$$(\underset{\sum_j B_j(\ )B_j^\dagger}{\mathcal{F}} \circ \underset{\sum_i A_i(\ )A_i^\dagger}{\mathcal{E}})(\rho) = \mathcal{F}(\mathcal{E}(\rho)) = \mathcal{F}\left(\sum_i A_i \rho A_i^\dagger\right)$$

$$= \sum_{ij} \overbrace{B_j A_i} \rho A_i^\dagger B_j^\dagger$$

Can introduce new Kraus operators $\rightarrow C_{\underbrace{ij}_{\equiv k}}$
(for the concatenated channel)

$$= \sum_k C_k \rho C_k^\dagger$$

Check: $\quad \sum_k C_k^\dagger C_k = \sum_{ij} A_i^\dagger B_j^\dagger B_j A_i = \sum_i A_i^\dagger \overbrace{\sum_j B_j^\dagger B_j}^{I} A_i = I \quad ✓$

Punchline: Basically everything that happens in quantum theory can be thought of as a quantum channel!

- State preparation
- Evolution
- Discarding a system
- Measurement
- Updates after measurement
- Interactions with an environment

more on this to come

\$ This unifying perspective, as we will see, can at times be helpful.